Journal of Mathematical Analysis and Applications **257,** 100–130 (2001)
doi:10.1006/jmaa.2000.7324, available online at http://www.idealibrary.com on IDEAL®

Existence Results for Some Nonuniformly Elliptic Equations with Irregular Data

Daniela Giachetti

E-mail: giachett@dmmm.uniroma1.it

and

Maria Michaela Porzio

Facoltá di Scienze Matematiche, Fisiche e Naturali, Universitá degli Studi del Sannio, via Port'Arsa 11, 82100 Benevento, Italy
E-mail: porzio@mat.uniroma1.it

Submitted by L. A. Peletier

Received March 23, 2000

In this paper we prove the existence of solutions of some nonlinear elliptic equations with degenerate coercivity of the type

$$\begin{cases} -\operatorname{div}\big(Du/(1+|u|)^{\theta}\big) = -\operatorname{div}(F), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

when the datum F belongs to the space $L^r(\Omega)$, where $r \leq 2$ and $\theta \in (0,1)$.

Key Words: existence and regularity; irregular data; nonlinear elliptic equations.

INTRODUCTION

In this paper we prove existence of weak solutions of elliptic equations of the type

$$\begin{cases} -\operatorname{div}(a(x,u)\,Du) = -\operatorname{div}(F), & \text{in } D'(\Omega), \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

0022-247X/01 $35.00

Here Ω is a bounded regular open set in $\mathbf{R}^N$, $N \geq 1$, $a(x,s)$ is a Carathéodory function defined in $\Omega \times \mathbf{R}$ with values in $\mathbf{R}$, satisfying

$$\begin{gathered} \beta \geq a(x,s) \geq \frac{\alpha}{(1+|s|)^{\theta}}, \qquad \text{a.e. } x \in \Omega \text{ and } \forall s \in \mathbf{R} \\ 0 < \theta < 1, \qquad \alpha, \beta > 0, \end{gathered} \tag{2}$$

and F verifies

$$F \in L^r(\Omega), \qquad \text{for some } r \in [r_0, 2], \tag{3}$$

where r_0 is a suitable constant that depends only on α, β, N, and Ω.

Let us point out that (2) implies that the coercivity in $H_0^1(\Omega)$ can degenerate when u is large.

Moreover, by (3) the right-hand side can belong to a space which is larger than $H^{-1}(\Omega)$ which would be the natural one for the problem. Therefore standard methods of uniformly elliptic operators acting between spaces in duality cannot be applied.

Our main results are stated in Section 1. In Theorems 1.1, 1.2, and 1.3 we consider data F in $L^2(\Omega)$.

If $0 < \theta \leq \frac{N}{2(N-1)}$ we prove existence of solutions in $W_0^{1,q}(\Omega)$, where

$$q = \begin{cases} \dfrac{2N(1-\theta)}{N-2\theta}, & \text{if } N > 2, \\ 2-\epsilon, \forall \epsilon \in (0,1], & \text{if } N = 2, \\ 2, & \text{if } N = 1 \end{cases}$$

(Theorem 1.1). We point out that $q < 2$ if $N \geq 2$ and therefore the solutions do not have "finite energy."

If $\theta > \frac{N}{2(N-1)}$ we prove existence of entropy solutions for problem (1) (see Definition 1.1 and Theorem 1.2).

The general case when the principal part is possibly nonlinear in the gradient is still open.

Then we consider less regular data $F \in L^r(\Omega)$, $1 < r < 2$, in Theorems 1.4 and 1.5.

More precisely Theorem 1.4 deals with the problems of the type (1) under the stronger control from above (with respect (2))

$$a(x,s) \leq \frac{\gamma}{(1+|s|)^{\theta}}, \qquad \text{a.e. } x \in \Omega, \text{ and } \forall s \in \mathbf{R}. \tag{4}$$

An example of $a(x,s)$ which doesn't satisfy (4) but simply (2) is

$$a(x,s) = \frac{1}{(1+\alpha(x)|s|)^{\theta}}, \tag{5}$$

where

$$0 \leq \alpha(x) \leq 1, \qquad \text{a.e. } x \in \Omega.$$

In Theorem 1.5 we deal with nonlinearity $a(x, s)$ satisfying assumption (2) and a further regularity condition (see (19) below).

Existence results for nonuniformly elliptic equations of the type (1) with the right-hand side in Lebesgue spaces have been proved in [3] with really different techniques, which cannot applied in our framework (see Remark 1.1).

On the other hand, existence of solutions for problems (1) when $F \in L^r(\Omega)$, with suitable r near two, are known only for uniformly elliptic operators ($\theta = 0$). Results for linear operators has been proved by Meyers (see [8]) just in the sixties, by means of the duality method and the regularity theorem. In the nonlinear setting there are some existence results for problems with right-hand side F zero and non-zero boundary conditions (see [6]). If the datum F is non-zero and the operator is strictly monotone and Lipschitz continuous in the gradient variable, again it is possible to prove the existence of solutions (see [2]).

Here the main difficulties (see Theorems 1.4 and 1.5) rely on the a priori estimates of Du_n (u_n is a suitable approximating sequence) in some $L^q(\Omega)$ space for $q < 2$. This is due to the lack of coercivity and the fact that when the data are not regular, we cannot use the natural test functions since they do not have the right summability.

To prove such estimates we will use some technical tools like a "decomposition" lemma and the Hodge decomposition that will be stated in Section 2. An obvious consequence of these a priori estimates will be the weak convergence of the gradients (up to a subsequence) that in the case of linear dependence of the gradient is sufficient to pass to the limit in the approximating problems.

1. MAIN RESULTS

Let us consider the following nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,u)\, Du) = -\operatorname{div}(F), & \text{in } D'(\Omega), \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Here Ω is a bounded open subset of $\mathbf{R}^{\mathbf{N}}$ and $a(x, s)\colon \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function satisfying, for every $s \in \mathbf{R}$ and for almost $x \in \Omega$, the structure conditions

$$\frac{c_0}{(1+|s|)^{\theta}} \le a(x,s) \le c_1, \tag{7}$$

$$0 < \theta < 1, \tag{8}$$

where c_0 and c_1 are positive constants. Let us begin by considering the more regular case when the datum F satisfies

$$F \in L^2(\Omega). \tag{9}$$

THEOREM 1.1. *Let* (7)–(9) *hold. Moreover if* $N > 2$ *let* $\theta \le \frac{N}{2(N-1)}$. *Then there exists at least a weak solution* $u \in W_0^{1,q}(\Omega)$ *of problem* (6) *in the sense of distribution, where*

$$q = \begin{cases} \dfrac{2N(1-\theta)}{N-2\theta}, & \textit{if } N > 2, \\ 2-\epsilon, \forall\epsilon \in (0,1], & \textit{if } N = 2, \\ 2, & \textit{if } N = 1. \end{cases} \tag{10}$$

Moreover it results

$$\int_\Omega \frac{|Du|^2}{(1+|u|)^{2\theta}} \le c, \tag{11}$$

where c is a constant that depends only on the data, i.e., on $N, \Omega, \theta, c_0, c_1$ *and* $\|F\|_2$.

Remark 1.1. We notice that our datum $-\operatorname{div}(F)$ belongs to the space $H^{-1}(\Omega)$ and that, if $N > 2$, $H^{-1}(\Omega) \supset L^{\gamma}(\Omega)$, where $\gamma = \frac{2N}{N+2}$. Thus Theorem 1.1 gives, as a particular case, an existence result of solutions belonging to the space $W_0^{1,q}(\Omega)$ (q as in (10)) when the datum is a function of $L^{\gamma}(\Omega)$, that is the same proved in [3] (with the same value of q) when $N > 2$.

When $N > 2$ the assumption $\theta \le \frac{N}{2(N-1)}$ guarantees that $q \ge 1$. If $\theta > \frac{N}{2(N-1)}$ it is possible to prove the existence of an entropy solution in the sense made precise below.

Let u be a measurable function such that $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$. We recall that there exists a unique measurable function $v : \Omega \to \mathbf{R}^{\mathbf{N}}$ such that

$$v\chi_{\{|u|<k\}} = DT_k(u), \qquad \text{almost everywhere in } \Omega, \forall k > 0,$$

(see [1, Lemma 2.1]). We define Du, the weak gradient of u, as this function v.

DEFINITION 1.1. A measurable function u is an entropy solution of (6) if $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$ and if

$$\int_\Omega a(x,u)\, Du\, DT_k(u-\varphi)\, dx \le \int_\Omega F\, DT_k(u-\varphi)\, dx,$$

for every $k > 0$ and for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

THEOREM 1.2. *Let us assume* (7), (9),

$$N > 2 \qquad \text{and} \qquad 1 > \theta > \frac{N}{2(N-1)}. \tag{12}$$

Then there exists an entropy solution u of problem (6). *Moreover such a solution satisfies*

$$\int_\Omega \frac{|Du|^2}{(1+|u|)^{2\theta}} \le c, \tag{13}$$

where c is a constant that depends only on the data.

We point out that in the simpler case where a has the following upper growth condition

$$\frac{c_0}{(1+|s|)^\theta} \le a(x,s) \le \frac{c_1}{(1+|s|)^\theta}, \qquad c_1 > 0, \tag{14}$$

the condition (11) or (13) is equivalent to

$$\int_\Omega a(x,u)^2 |Du|^2 \le c_2, \tag{15}$$

where $c_2 = cc_1$. Hence we can take test functions in $H_0^1(\Omega)$.

Examples of operators verifying (14) are

$$a(x,s) = \frac{\alpha(x)}{(1+|s|)^\theta}, \tag{16}$$

or

$$a(x,s) = \frac{1}{(\alpha(x)+|s|)^\theta}, \tag{17}$$

where α satisfies, for almost every $x \in \Omega$, the conditions

$$\alpha \in L^\infty(\Omega), \qquad \alpha(x) \ge c_2 > 0. \tag{18}$$

We notice that in the case (16) as in the case (17) the condition (18) on α is a necessary and sufficient condition for (14) to hold.

There exist examples of operators for which condition (7) is satisfied but (14) is violated, as shown in the introduction by the operator defined in (5).

Nevertheless even in this case if we assume further conditions, like, for example, if $N > 2$, that $\frac{|D\alpha(x)|}{\alpha(x)} \in L^N(\Omega)$, then the regularity condition (15) still holds true. More generally we have the following

THEOREM 1.3. *Under the assumptions* (7)–(9) *if it results*

$$G(x) \equiv \sup_{s \in \mathbf{R}} \frac{|D_x(a(x,s))|}{a(x,s)} \in L^m(\Omega),$$

$$m = \begin{cases} N, & \text{if } N > 2, \\ 2 + \nu, & \text{if } N = 2, \\ 2, & \text{if } N = 1, \end{cases} \tag{19}$$

where ν is a positive constant, then the solutions u found in Theorems 1.1 *and* 1.2 *verify*

$$\int_\Omega a(x,u)^2 |Du|^2 \leq c_3, \tag{20}$$

where c_3 is a constant that depends only on the data. Hence we can take test functions in $H_0^1(\Omega)$.

Consider now the case when F satisfies the weaker condition

$$F \in L^r(\Omega), \qquad \frac{N}{N(1-\theta)+\theta} < r < 2, \tag{21}$$

and

$$N = 1, \qquad \text{or} \qquad N > 2 \qquad \text{and} \qquad \theta < \frac{N}{2(N-1)}. \tag{22}$$

Notice that assumption (22) guarantees that the hypothesis (21) is well defined as it is equivalent to the inequality $\frac{N}{N(1-\theta)+\theta} < 2$.

In this case, as before, we treat the two different upper growth conditions (7) and (14) as shown by the following Theorems 1.4 and 1.5.

THEOREM 1.4. *Let θ be as in* (22) *and assume that* (14) *and* (21) *hold. There exists a constant r_0 ($r_0 < 2$) that depends only on N, Ω, c_0, and c_1 such that if r in* (21) *satisfies*

$$r_0 \leq r < 2, \tag{23}$$

then there exists at least a solution $u \in W_0^{1,q}(\Omega)$ of (6), *where q is given by the formula*

$$q = \begin{cases} \dfrac{Nr(1-\theta)}{N-\theta r}, & \text{if } N \geq 2, \\ r, & \text{if } N = 1. \end{cases} \tag{24}$$

Remark 1.2. We notice that the coefficient q in (24) verifies the condition $q > 1$. As a matter of fact it is obvious if $N = 1$; otherwise such an inequality is equivalent to the condition (21), $r > \frac{N}{N(1-\theta)+\theta}$.

THEOREM 1.5. *Let θ be as in* (22). *Under the assumption* (7) *if $G \in L^{\sigma}(\Omega)$ where G is as in* (19) *and $\sigma = \max\{N, 2\}$, there exists a constant r_1 ($r_1 < 2$) that depends only on N, c_0, c_1, and Ω such that if r in* (21) *satisfies*

$$r_1 \le r < 2, \tag{25}$$

then there exists at least one solution $u \in W_0^{1,q}(\Omega)$ of problem (6), *where q is as in* (24).

2. SOME PRELIMINARY RESULTS

As said above, we state here the Hodge decomposition and a decomposition lemma that will be used in the following.

LEMMA 2.1 (Hodge Decomposition). *Let Ω be a regular domain (for the definition see* [6]), *$v \in W_0^{1,r}(\Omega)$, $r > 1$, and let $-1 < \epsilon < r - 1$. Then there exist ϕ: $\Omega \to \mathbf{R}$ and H: $\Omega \to \mathbf{R}^{\mathbf{N}}$ such that $H \in (L^{r/(1+\varepsilon)}(\Omega))^N$, $divH = 0$, $\phi \in W_0^{1, r/(1+\varepsilon)}(\Omega)$, and*

$$|Dv|^{\epsilon} Dv = D\phi + H, \tag{26}$$

$$\|H\|_{L^{r/(1+\epsilon)}(\Omega)} \le c(\Omega, r, N)|\epsilon| \cdot \|Dv\|_{L^r(\Omega)}^{1+\epsilon}, \tag{27}$$

$$\|D\phi\|_{L^{r/(1+\epsilon)}(\Omega)} \le (1 + c(\Omega, r, N)|\epsilon|)\|Dv\|_{L^r(\Omega)}^{1+\epsilon}. \tag{28}$$

For the proof see [5, 6].

LEMMA 2.2 (Decomposition Lemma). *Let $G \in L^m(\Omega)$, $v \in W_0^{1,r}(\Omega)$ where $r \in (1, +\infty)$ and $\varepsilon \in \mathbf{R}^+$. Then there exist $\Omega_1, \Omega_2, \dots, \Omega_l$ measurable subsets of Ω and l elements v_s of $W_0^{1,r}(\Omega)$, $s \in \{1, \dots, l\}$ such that*

$$\|G\|_{L^m(\Omega_l)} \le \varepsilon, \qquad \|G\|_{L^m(\Omega_s)} = \varepsilon, \qquad s = \{1, 2, \dots, l-1\}, \tag{29}$$

$$l \le \left(\varepsilon^{-1}\|G\|_{L^m(\Omega)}\right)^m + 1, \tag{30}$$

$$\left\{x \in \Omega : |D(v_s)| \ne 0\right\} \subset \Omega_s, \qquad s \in \{1, 2, \dots, l\}, \tag{31}$$

$$Dv = Dv_s \qquad \text{a.e. in } \Omega_s, \qquad s \in \{1, 2, \dots, l\}, \tag{32}$$

$$v_1 + v_2 + \cdots + v_l = v \qquad \text{in } \Omega, \tag{33}$$

$$v = \sum_{i=s}^{l} v_i \qquad \text{in } \Omega_s. \tag{34}$$

The proof of (29)–(33) can be found in [4, Proposition 2.1], while the property (34), that will be essential in the a priori estimates, will be easily proved once the construction in [4] is seen.

3. A PRIORI ESTIMATES

In this section we will prove some a priori estimates that will be essential in the proofs of Theorems 1.1–1.5. These will be true also in the case of more general nonlinear operators (also with respect to the gradient variable) under classical hypotheses. We confine ourselves to the case where the operator is linear in the gradient variable, since at the moment, we are not able to pass to the limit in the approximating problems when the operator is not of this type.

3.1. *First Case*: *Datum in* $H^{-1}(\Omega)$

Let us consider the following approximating problems

$$\begin{cases} -\operatorname{div}\big(a(x, T_n(u_n))\, Du_n\big) = -\operatorname{div}(F), & \text{in } \Omega, \\ u_n \in H_0^1(\Omega), \end{cases} \tag{35}$$

where $T_n(s)$ is the usual truncation at levels $\pm n$, that is,

$$T_n(s) = \max\{-n, \min\{n, s\}\}. \tag{36}$$

We notice that under the hypotheses of Theorem 1.1 and Theorem 1.2 for every fixed n in N, there exists one and only one solution of the approximating problem in (35).

LEMMA 3.3. *Under the assumptions of Theorem* 1.1 *there exists a constant* c_4 *that depends only on* N, q, Ω, c_0, *and* $\|F\|_{L^2(\Omega)}$ *such that every solution of problem* (35) *satisfies the estimate*

$$\int_\Omega |Du_n|^q \le c_4, \tag{37}$$

where q *is as in* (10).

Proof. As a first step we prove that, for every fixed θ, $0 < \theta < 1$, there exists a constant c_5, that doesn't depend on n, such that it results

$$\int_\Omega \frac{|Du_n|^2}{\big(1 + |T_n(u_n)|\big)^{2\theta}} \le c_5, \tag{38}$$

for every solution u_n of (35). Let us define

$$\psi(u_n) = \int_0^{u_n} \frac{ds}{\big(1 + |T_n(s)|\big)^{\theta}}. \tag{39}$$

It results $\psi(u_n) \in H_0^1(\Omega)$ and we have

$$D\psi(u_n) = \frac{Du_n}{(1+|T_n(u_n)|)^\theta}. \tag{40}$$

Taking $\psi(u_n)$ as a test function in (35) and by means of assumption (7) and the Young inequality we obtain

$$\begin{aligned} \int_\Omega \frac{c_0|Du_n|^2}{(1+|T_n(u_n)|)^{2\theta}} &\le \int_\Omega a(x,T_n(u_n))\frac{|Du_n|^2}{(1+|T_n(u_n)|)^\theta} \\ &\le \int_\Omega \frac{c_0|Du_n|^2}{2(1+|T_n(u_n)|)^{2\theta}} + c_6, \end{aligned} \tag{41}$$

where $c_6 = (1/c_0)\int_\Omega |F|^2$, which implies (38) with $c_5 = 2c_6/c_0$. To conclude the proof we notice that when $N \ge 2$ it results $q < 2$, and so using the Sobolev and the Hoelder inequality we have

$$\begin{aligned} c_s^{-q}\left(\int_\Omega |u_n|^{q^*}\right)^{q/q^*} &\le \int_\Omega |Du_n|^q \\ &= \int_\Omega \frac{|Du_n|^q}{(1+|T_n(u_n)|)^{\theta q}}(1+|T_n(u_n)|)^{\theta q} \\ &\le \left(\int_\Omega \frac{|Du_n|^2}{(1+|T_n(u_n)|)^{2\theta}}\right)^{q/2}\left(\int_\Omega (1+|u_n|)^{\theta q(2/q)'}\right)^{1-q/2}, \end{aligned} \tag{42}$$

where c_s is the Sobolev constant.[1] From (42), using the estimate (38) we obtain that if $N \ge 2$,

$$\begin{aligned} c_s^{-q}\left(\int_\Omega |u_n|^{q^*}\right)^{q/q^*} &\le \int_\Omega |Du_n|^q \\ &\le (c_5)^{q/2}\left(\int_\Omega (1+|u_n|)^{\theta q(2/q)'}\right)^{1-q/2}. \end{aligned} \tag{43}$$

If $N > 2$ observing that $q^* = \theta q(2/q)'$ and that $1 - q/2 < q/q^*$ from (43) it follows

$$\int_\Omega |u_n|^{q^*} \le c_7, \tag{44}$$

[1] I.e., $c_s = c(\Omega, N)$ such that for every $\omega \in W_0^{1,p}(\Omega)$ it results $\|\omega\|_{L^\tau(\Omega)} \le c_s\|D\omega\|_{L^p(\Omega)}$, where $\tau = p^*$, if $N > p$, τ can be any real number bigger than one if $N = p$, and $\tau = +\infty$, if $N < p$.

where $c_7 = (c_5^{q/2} c_s^q)^{1/(q/q^* - 1 + q/2)}$. From (44) and (43) it follows (37).

If $N = 2$, it results $\theta q(\frac{2}{q})' < q^*$ and thus using the Hoelder estimate we obtain

$$\left(\int_\Omega (1 + |u_n|)^{\theta q(2/q)'} \right)^{1-q/2} \leq c_8 \left(\int_\Omega (1 + |u_n|)^{q^*} \right)^{\theta(N-q)/N}, \quad (45)$$

where $c_8 = c(|\Omega|, \theta, q, N)$. From (43) and (45) noticing that $\theta(N - q)/N < q/q^*$ we obtain (44), that again united to (43) gives (37).

Finally, if $N = 1$ by the Sobolev immersion theorem it results $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and thus, using the estimate (38), we have

$$\int_\Omega |Du_n|^2 = \int_\Omega \frac{|Du_n|^2}{(1 + |T_n(u_n)|)^{2\theta}} (1 + |T_n(u_n)|)^{2\theta}$$

$$\leq c_5 \| 1 + |T_n(u_n)| \|_{L^\infty(\Omega)}^{2\theta} \leq 2c_9 \left(1 + \|Du_n\|_{L^2(\Omega)}^{2\theta} \right), \quad (46)$$

where $c_9 = c_5 \max\{1, c_s^2\}$. From (46) follows the thesis as $\theta < 1$. Q.E.D.

LEMMA 3.4. *Under the assumptions of Theorem* 1.3 *there exists a constant* c_{10} *that depends only on* N, Ω, c_0, *and* $\|F\|_{L^2(\Omega)}$ *such that every solution of problem* (35) *satisfies the estimate*

$$\int_\Omega a(x, T_n(u_n))^2 |Du_n|^2 \leq c_{10}. \quad (47)$$

Proof. Let $\psi(u_n)$ be the function defined as

$$\psi(u_n) = \int_0^{u_n} a(x, T_n(s))\, ds. \quad (48)$$

It results $\psi(u_n) \in H_0^1(\Omega)$ and

$$D\psi(u_n) = a(x, T_n(u_n))\, Du_n + \int_0^{u_n} D_x a(x, T_n(s))\, ds. \quad (49)$$

To prove (47) we first show that the following estimate holds

$$\int_\Omega |D\psi(u_n)|^2 \leq c_{11}. \quad (50)$$

Let ε be a positive constant that will be determined later (depending only on the data in the structure conditions) and apply Lemma 2.2 with $v = \psi(u_n)$ and $r = 2$. Thus there exist $\Omega_1, \Omega_2, \dots, \Omega_l$ measurable subsets of Ω and l elements $v_s = [\psi(u_n)]_s$ of $H_0^1(\Omega)$, $s \in \{1, \dots, l\}$, such that the conditions (29)–(34) are satisfied. Let $s \in \{1, \dots, l\}$ be arbitrary fixed and choose $[\psi(u_n)]_s$ as a test function in (35). We obtain

$$\int_\Omega a(x, T_n(u_n))\, Du_n\, D[\psi(u_n)]_s = \int_\Omega F D[\psi(u_n)]_s. \quad (51)$$

We estimate now the integrals that appear in (51). Using (49), the structure assumptions, and Lemma 2.2 it results

$$
\begin{aligned}
&\int_{\Omega} a(x, T_n(u_n))\, Du_n D[\psi(u_n)]_s \\
&\qquad = \int_{\Omega_s} [a(x, T_n(u_n))]^2 |Du_n|^2 \\
&\qquad\qquad + \int_{\Omega_s} a(x, T_n(u_n))\, Du_n \left(\int_0^{u_n} D_x a(x, T_n(s))\, ds \right) dx \\
&\qquad \geq \int_{\Omega_s} [a(x, T_n(u_n))]^2 |Du_n|^2 \\
&\qquad\qquad - \int_{\Omega_s} a(x, T_n(u_n)) |Du_n| \left(\int_0^{|u_n|} a(x, T_n(s)) \frac{|D_x a(x, T_n(s))|}{a(x, T_n(s))}\, ds \right) \\
&\qquad \geq \int_{\Omega_s} [a(x, T_n(u_n))]^2 |Du_n|^2 - \int_{\Omega_s} a(x, T_n(u_n)) |Du_n| \, |\psi(u_n)| G(x) \\
&\qquad \geq \int_{\Omega_s} [a(x, T_n(u_n))]^2 |Du_n|^2 - \epsilon_1 \int_{\Omega_s} [a(x, T_n(u_n))]^2 |Du_n|^2 \\
&\qquad\qquad - c_{12} \int_{\Omega_s} G(x)^2 |\psi(u_n)|^2,
\end{aligned}
\tag{52}
$$

where ϵ_1 is a positive constant that will be chosen later and $c_{12} = (\epsilon_1)^{-1}$. Besides for the right hand side of (51) we have

$$
\begin{aligned}
&\int_{\Omega} F D[\psi(u_n)]_s \\
&\qquad = \int_{\Omega_s} F \left[a(x, T_n(u_n))\, Du_n + \int_0^{u_n} a(x, T_n(s)) \frac{D_x a(x, T_n(s))}{a(x, T_n(s))}\, ds \right] \\
&\qquad \leq \epsilon_2 \int_{\Omega_s} [a(x, T_n(u_n))]^2 |Du_n|^2 + c_{13} \int_{\Omega} |F|^2 + \int_{\Omega_s} |F| G(x) |\psi(u_n)| \\
&\qquad \leq \epsilon_2 \int_{\Omega_s} [a(x, T_n(u_n))]^2 |Du_n|^2 + (c_{13} + 1) \int_{\Omega} |F|^2 \\
&\qquad\qquad + \int_{\Omega_s} [G(x)]^2 |\psi(u_n)|^2,
\end{aligned}
\tag{53}
$$

where ϵ_2 is a positive constant that will be chosen later and $c_{13} = (\epsilon_2)^{-1}$. Using (52) and (53) in (51) we obtain

$$(1 - \epsilon_1 - \epsilon_2)\int_{\Omega_s} [a(x, T_n(u_n))]^2 |Du_n|^2$$
$$\leq (c_{13} + 1)\int_{\Omega} |F|^2 + (c_{12} + 1)\int_{\Omega_s} [G(x)]^2 |\psi(u_n)|^2. \quad (54)$$

Choosing ϵ_1 and ϵ_2 in (54) as

$$\epsilon_1 = \epsilon_2 = \tfrac{1}{4}, \quad (55)$$

it holds

$$\tfrac{1}{2}\int_{\Omega_s} [a(x, T_n(u_n))]^2 |Du_n|^2 \leq c_{14}\int_{\Omega} |F|^2 + c_{15}\int_{\Omega_s} [G(x)]^2 |\psi(u_n)|^2, \quad (56)$$

where $c_{14} = c_{13} + 1$ and $c_{15} = c_{12} + 1$. We notice now that using again Lemma 2.2 and the Hoelder and Sobolev inequalities, we can estimate the last term in (56) as

$$\begin{aligned}\int_{\Omega_s} [G(x)]^2 |\psi(u_n)|^2 &\leq \sum_{i=s}^{l} \int_{\Omega_s} [G(x)]^2 |[\psi(u_n)]_i|^2 \\ &\leq \sum_{i=s}^{l} \|G\|^2_{L^m(\Omega_s)} \|[\psi(u_n)]_i\|^2_{L^\gamma(\Omega)} \\ &\leq c_s^2 \sum_{i=s}^{l} \|G\|^2_{L^m(\Omega_s)} \int_{\Omega} |D[\psi(u_n)]_i|^2, \quad (57)\end{aligned}$$

where m is as in (19), c_s is the Sobolev immersion constant, and

$$\gamma = \begin{cases} 2^*, & \text{if } N > 2, \\ 2\left(\dfrac{m}{2}\right)', & \text{if } N = 2, \\ +\infty, & \text{if } N = 1. \end{cases}$$

Besides using (56), (57), and (29) we have

$$\begin{aligned}\int_\Omega \left|D[\psi(u_n)]_s\right|^2 &= \int_{\Omega_s}\left|a(x,T_n(u_n))\,Du_n + \int_0^{u_n} D_x a(x,T_n(s))\,ds\right|^2\\
&\le 2\int_{\Omega_s}\left[a(x,T_n(u_n))\right]^2 |Du_n|^2\\
&\quad + 2\int_{\Omega_s}\left|\int_0^{u_n} a(x,T_n(s))\frac{D_x a(x,T_n(s))\,ds}{a(x,T_n(s))}\right|^2\\
&\le c_{16} + c_{17}\int_{\Omega_s}[G(x)]^2 |\psi(u_n)|^2\\
&\le c_{16} + c_{18}\sum_{i=s}^{l}\|G\|^2_{L^m(\Omega_s)}\int_\Omega \left|D[\psi(u_n)]_i\right|^2\\
&\le c_{16} + c_{18}\varepsilon^2\sum_{i=s}^{l}\int_\Omega \left|D[\psi(u_n)]_i\right|^2, \qquad (58)\end{aligned}$$

where $c_{16} = 4c_{14}\int_\Omega |F|^2$, $c_{17} = 4c_{15} + 2$, and $c_{18} = c_{17}c_s^2$. Let $s = l$ in (58); we obtain

$$\left(1 - c_{18}\varepsilon^2\right)\int_\Omega \left|D[\psi(u_n)]_l\right|^2 \le c_{16}, \qquad (59)$$

from which it follows, choosing $\varepsilon = (2c_{18})^{-1/2}$,

$$\int_\Omega \left|D[\psi(u_n)]_l\right|^2 \le c_{19}, \qquad (60)$$

where $c_{19} = 2c_{16}$. Let now $s = l - 1$ in (58); we have

$$\begin{aligned}&\int_\Omega \left|D[\psi(u_n)]_{l-1}\right|^2\\
&\qquad \le c_{16} + c_{18}\varepsilon^2\left(\int_\Omega \left|D[\psi(u_n)]_{l-1}\right|^2 + \int_\Omega \left|D[\psi(u_n)]_l\right|^2\right), \qquad (61)\end{aligned}$$

that together with (60), thanks to the choice done for ε, implies

$$\int_\Omega \left|D[\psi(u_n)]_{l-1}\right|^2 \le c_{20}, \qquad (62)$$

where $c_{20} = 2c_{16} + c_{19}$. Proceeding as above, we deduce that for every $s \in \{1, \dots, l\}$ it results

$$\int_\Omega \left|D[\psi(u_n)]_s\right|^2 \le c_{21}, \qquad (63)$$

where c_{21} is a constant that depends only on the data in the structure conditions. From (33), (63), and (30) it follows that

$$\int_\Omega |D\psi(u_n)|^2 \le c_{22}, \tag{64}$$

where $c_{22} = c_{21}[(\varepsilon^{-1}\|G\|_{L^m(\Omega)})^m + 1]^2$, that is, (50). Since we can easily verify that

$$\begin{aligned}\int_\Omega \left|\int_0^{u_n} D_x a(x, T_n(s))\, ds\right|^2 &\le \int_\Omega [G(x)]^2 \left|\int_0^{|u_n|} a(x, T_n(s))\, ds\right|^2 \\ &= \int_\Omega [G(x)]^2 |\psi(u_n)|^2 \\ &\le \|G\|_{L^m(\Omega)} c_{23} \int_\Omega |D\psi(u_n)|^2,\end{aligned} \tag{65}$$

where $c_{23} = c_s^2$, (50) implies (47) due to (49). Q.E.D.

Remark 3.3. We notice that we can avoid using Lemma 2.2 if we know that $\|G\|_{L^m(\Omega)}$ is sufficiently small.

3.2. *Second Case*: *Datum Not in* $H^{-1}(\Omega)$

Let us consider the following approximating problems

$$\begin{cases} -\operatorname{div}(a(x, T_n(u_n))\, Du_n) = -\operatorname{div}(F_n), & \text{in } \Omega, \\ u_n \in H_0^1(\Omega), \end{cases} \tag{66}$$

where F_n is a sequence in $L^2(\Omega)$ such that

$$F_n \to F, \qquad \text{in } L^r(\Omega). \tag{67}$$

It is well known that if $a(x, s)$ is as in Theorems 1.4 or 1.5 then, for every fixed $n \in \mathbf{N}$, there exists one and only one solution of (66). We begin with the simpler case when we have the strong control from above stated in (14).

LEMMA 3.5. *Under the hypotheses of Theorem* 1.4 *there exists a constant* c_{25} *that depends only on* c_0, c_1, N, *and* Ω *such that the following estimate holds*

$$\int_\Omega \frac{|Du_n|^r}{\left(1+|T_n(u_n)|\right)^{\theta r}} \le c_{25}, \tag{68}$$

where u_n *is the solution of problem* (66).

Proof. Let $v = \psi(u_n)$ be as in (39) and $\phi(u_n)$ and H_n defined by Lemma 2.1 where we choose $\epsilon = r - 2$. Hence $\phi(u_n) \in W_0^{1,(r/(r-1))}(\Omega)$ and it results

$$D\phi(u_n) = |D\psi(u_n)|^{r-2} D\psi(u_n) - H_n, \tag{69}$$

where

$$\|H_n\|_{L^{r/(r-1)}(\Omega)} \le c(\Omega, N)|r-2| \cdot \|D\psi(u_n)\|_{L^r(\Omega)}^{r-1}. \tag{70}$$

Choose $\phi(u_n)$ as a test function in (66). Using (69) we obtain

$$\begin{aligned} &\int_\Omega a(x, T_n(u_n))\, Du_n\left[|D\psi(u_n)|^{r-2} D\psi(u_n) - H_n\right] \\ &\qquad = \int_\Omega F_n\left[|D\psi(u_n)|^{r-2} D\psi(u_n) - H_n\right]. \end{aligned} \tag{71}$$

Using (40) and (14) we have

$$\begin{aligned} &\int_\Omega a(x, T_n(u_n))\, Du_n |D\psi(u_n)|^{r-2} D\psi(u_n) \\ &\qquad = \int_\Omega \frac{a(x, T_n(u_n))|Du_n|^r}{\left(1+|T_n(u_n)|\right)^{\theta(r-1)}} \\ &\qquad \ge c_0 \int_\Omega \frac{|Du_n|^r}{\left(1+|T_n(u_n)|\right)^{\theta r}}, \end{aligned} \tag{72}$$

that with (71) implies

$$\begin{aligned} c_0 \int_\Omega \frac{|Du_n|^r}{\left(1+|T_n(u_n)|\right)^{\theta r}} &\le \int_\Omega a(x, T_n(u_n))|Du_n|\,|H_n| \\ &\qquad + \int_\Omega |F_n|\left[|D\psi(u_n)|^{r-1} + |H_n|\right]. \end{aligned} \tag{73}$$

We estimate now the terms in the right-hand side of (73). Using assumption (14), (70), and (40) it results

$$\begin{aligned}
&\int_\Omega a(x, T_n(u_n))|Du_n|\,|H_n| \\
&\quad \le c_1 \int_\Omega \frac{|Du_n|}{(1+|T_n(u_n)|)^\theta}|H_n| \\
&\quad \le c_1\epsilon_1 \int_\Omega \frac{|Du_n|^r}{(1+|T_n(u_n)|)^{\theta r}} + c_1 c(\epsilon_1)\int_\Omega |H_n|^{r/(r-1)} \\
&\quad \le c_1\epsilon_1 \int_\Omega \frac{|Du_n|^r}{(1+|T_n(u_n)|)^{\theta r}} \\
&\qquad + c_1 c(\epsilon_1)(c(\Omega, N)|r-2|)^{r/(r-1)}\int_\Omega |D\psi(u_n)|^r \\
&\quad = c_1\Big(\epsilon_1 + c(\epsilon_1)(c(\Omega,N)|r-2|)^{r/(r-1)}\Big)\int_\Omega \frac{|Du_n|^r}{(1+|T_n(u_n)|)^{\theta r}}, \qquad (74)
\end{aligned}$$

where ϵ_1 is a positive constant to be determined and $c(\epsilon_1) = \epsilon_1^{1/(1-r)}$. Moreover, using again (40) we obtain

$$\begin{aligned}
\int_\Omega |F_n|\,|D\psi(u_n)|^{r-1} &\le \epsilon_2 \int_\Omega |D\psi(u_n)|^r + c(\epsilon_2)\int_\Omega |F_n|^r \\
&= \epsilon_2 \int_\Omega \frac{|Du_n|^r}{(1+|T_n(u_n)|)^{\theta r}} + c(\epsilon_2)\int_\Omega |F_n|^r, \qquad (75)
\end{aligned}$$

where ε_2 is a positive constant to be determined and $c(\epsilon_2) = \epsilon_2^{1-r}$. At the end, by (70) we also have

$$\int_\Omega |F_n|\,|H_n| \le \int_\Omega |F_n|^r + (c(\Omega, N)|r-2|)^{r/(r-1)}\int_\Omega \frac{|Du_n|^r}{(1+|T_n(u_n)|)^{\theta r}}. \qquad (76)$$

Using (74), (75), and (76) in (73) it follows that

$$\begin{aligned}
&\Big\{c_0 - \Big[c_1\epsilon_1 + (c_1 c(\epsilon_1) + 1)(c(\Omega, N)|r-2|)^{r/(r-1)} + \epsilon_2\Big]\Big\} \\
&\qquad \times \int_\Omega \frac{|Du_n|^r}{(1+|T_n(u_n)|)^{\theta r}} \le (c(\epsilon_2)+1)\int_\Omega |F_n|^r. \qquad (77)
\end{aligned}$$

We notice that (67) implies the existence of a positive constant c_{26} that depends only on $\|F\|_{L^r(\Omega)}$ and thus independent of n, such that

$$\int_\Omega |F_n|^r \le c_{26}. \tag{78}$$

Let us choose

$$c_1 \epsilon_1 = \epsilon_2 = \frac{c_0}{4}. \tag{79}$$

We want that r satisfies the inequality

$$\left[c_1\left(\frac{c_0}{4c_1}\right)^{1/(1-r)} + 1\right](c(\Omega, N)|r-2|)^{r/(r-1)} < \frac{c_0}{4},$$

and this is true if, for example, we choose r_0 verifying

$$|r_0 - 2| < \frac{c_0}{4c(\Omega, N)(c_1+1)}. \tag{80}$$

Using (78), (79), and (80) in (77) we obtain

$$\int_\Omega \frac{|Du_n|^r}{(1+|T_n(u_n)|)^{\theta r}} \le \frac{4}{c_0}\left[\left(\frac{c_0}{4}\right)^{1-r} + 1\right]c_{26} \le c_{27}, \tag{81}$$

where $c_{27} = \frac{4}{c_0}(\max\{1, \frac{4}{c_0}\} + 1)c_{26}$. Q.E.D.

We prove now the a priori estimate under the weaker condition (7).

LEMMA 3.6. *Under the hypotheses of Theorem* 1.5 *there exists a constant* c_{28} *that depends only on* c_0, N, $\|G\|_{L^\sigma(\Omega)}$, $\|F\|_{L^r(\Omega)}$, r, *and* Ω *such that the following estimate holds*

$$\int_\Omega [a(x, T_n(u_n))]^r |Du_n|^r \, dx \le c_{28}, \tag{82}$$

where u_n *is the solution of problem* (66).

Proof. Let $\psi(u_n)$ be as in (48) where now u_n is the solution of problem (66). Analogously to the proof of Lemma 3.4 we will prove that (82) is a consequence of the estimate

$$\int_\Omega |D\psi(u_n)|^r \leq c_{29}, \tag{83}$$

where c_{29} is a constant independent on n.

To do this, let

$$\varepsilon = \begin{cases} \dfrac{1}{16c_s}, & \text{if } N \geq 2, \\ \dfrac{1}{16c_s(1+|\Omega|)}, & \text{if } N = 1, \end{cases}$$

where c_s is the Sobolev immersion constant, and apply Lemma 2.2 with the functions $G \in L^\sigma(\Omega)$ and $v = \psi(u_n) \in W_0^{1,r}(\Omega)$. Then there exist $\Omega_1, \Omega_2, \ldots, \Omega_l$ measurable subsets of Ω and l elements $[\psi(u_n)]_s$ of $W_0^{1,r}(\Omega)$, $s \in \{1, \ldots, l\}$, such that the condition (29)–(34) are satisfied. Let $s \in \{1, \ldots, l\}$ be arbitrary fixed and let us apply Lemma 2.1 to the function $[\psi(u_n)]_s$. Thus we have, setting $\phi_n^{(s)}$ and $H_n^{(s)}$ the corresponding functions,

$$D\phi_n^{(s)} = |D[\psi(u_n)]_s|^{r-2} D[\psi(u_n)]_s - H_n^{(s)}, \tag{84}$$

where

$$\| H_n^{(s)} \|_{L^{r/(r-1)}(\Omega)} \leq c(\Omega, N)|r-2| \cdot \| D[\psi(u_n)]_s \|_{L^r(\Omega)}^{r-1}. \tag{85}$$

Using (49) and the fact that u_n is the solution of problem (66) we obtain

$$\begin{aligned} \int_\Omega D\psi(u_n)\, D\phi_n^{(s)} &= \int_\Omega a(x, T_n(u_n))\, Du_n\, D\phi_n^{(s)} \\ &\quad + \int_\Omega \left(\int_0^{u_n} D_x a(x, T_n(s))\, ds \right) D\phi_n^{(s)} \\ &= \int_\Omega F_n\, D\phi_n^{(s)} + \int_\Omega \left(\int_0^{u_n} D_x a(x, T_n(s))\, ds \right) D\phi_n^{(s)}. \end{aligned} \tag{86}$$

We notice that by (84) it results

$$\int_\Omega D\psi(u_n)\, D\phi_n^{(s)} = \int_\Omega |D[\psi(u_n)]_s|^r - \int_\Omega D\psi(u_n) H_n^{(s)}. \tag{87}$$

From (87) and (86) it follows that

$$\int_\Omega |D[\psi(u_n)]_s|^r \le \int_\Omega |D\psi(u_n)||H_n^{(s)}| + \int_\Omega |F_n\, D\phi_n^{(s)}| + \int_\Omega \left(\int_0^{|u_n|} |D_x a(x, T_n(s))|\, ds\right)|D\phi_n^{(s)}|. \tag{88}$$

We estimate now the integrals that appear in the right-hand side of (88). By means of the Young inequality we have

$$\int_\Omega |F_n\, D\phi_n^{(s)}| \le \varepsilon_1 \int_\Omega |D\phi_n^{(s)}|^{r/(r-1)} + \varepsilon_1^{1-r}\int_\Omega |F_n|^r. \tag{89}$$

where ε_1 is a positive constant that will be determined later. Moreover, using (84) and (85) we can estimate the first integral in the right-hand side of (89) as

$$\begin{aligned}\varepsilon_1 \int_\Omega |D\phi_n^{(s)}|^{r/(r-1)} &\le \varepsilon_1 \int_\Omega \left(|D[\psi(u_n)]_s|^{r-1} + |H_n^{(s)}|\right)^{r/(r-1)} \\ &\le \varepsilon_1 2^{r/(r-1)}\left[\int_\Omega |D[\psi(u_n)]_s|^r + \int_\Omega |H_n^{(s)}|^{r/(r-1)}\right] \\ &\le \varepsilon_1 2^{r/(r-1)} \int_\Omega |D[\psi(u_n)]_s|^r \\ &\quad + \varepsilon_1 (c_1(\Omega, N)|r-2|)^{r/(r-1)} \int_\Omega |D[\psi(u_n)]_s|^r,\end{aligned} \tag{90}$$

where $c_1(\Omega, N) = 2c(\Omega, N)$. From (90) and (89) we obtain

$$\int_\Omega |F_n\, D\phi_n^{(s)}| \le \varepsilon_1^{1-r}\int_\Omega |F_n|^r + \varepsilon_1 2^{r/(r-1)}\int_\Omega |D[\psi(u_n)]_s|^r + \varepsilon_1 (c_1(\Omega, N)|r-2|)^{r/(r-1)}\int_\Omega |D[\psi(u_n)]_s|^r.$$

We estimate now the first integral in the right-hand side of (88). Using Lemma 2.2 and the Young inequality it results

$$
\begin{aligned}
&\int_\Omega |D\psi(u_n)||H_n^{(s)}| \\
&\quad = \sum_{i=1}^{l} \int_{\Omega_i} |D[\psi(u_n)]_i||H_n^{(s)}| \\
&\quad \leq \sum_{i=1, i\neq s}^{l} \left(\int_{\Omega_i} |D[\psi(u_n)]_i|^r\right)^{1/r} \left(\int_\Omega |H_n^{(s)}|^{r/(r-1)}\right)^{(r-1)/r} \\
&\qquad + \left(\int_{\Omega_s} |D[\psi(u_n)]_s|^r\right)^{1/r} \left(\int_\Omega |H_n^{(s)}|^{r/(r-1)}\right)^{(r-1)/r} \\
&\quad \leq \varepsilon_2 \left[\sum_{i=1, i\neq s}^{l} \left(\int_{\Omega_i} |D[\psi(u_n)]_i|^r\right)^{1/r}\right]^r \\
&\qquad + c(\varepsilon_2)(c(\Omega, N)|r-2|)^{r/(r-1)} \int_\Omega |D[\psi(u_n)]_s|^r \\
&\qquad + \varepsilon_3 \int_\Omega |D[\psi(u_n)]_s|^r \\
&\qquad + c(\varepsilon_3)(c(\Omega, N)|r-2|)^{r/(r-1)} \int_\Omega |D[\psi(u_n)]_s|^r \\
&\quad \leq \varepsilon_2 (l-1)^r \sum_{i=1, i\neq s}^{l} \int_{\Omega_i} |D[\psi(u_n)]_i|^r \\
&\qquad + \left\{[c(\varepsilon_2) + c(\varepsilon_3)](c(\Omega, N)|r-2|)^{r/(r-1)} + \varepsilon_3\right\} \\
&\qquad \times \int_\Omega |D[\psi(u_n)]_s|^r,
\end{aligned}
$$

where ε_2 and ε_3 are positive constants to be determined and $c(\varepsilon_i) = \varepsilon_i^{1/(1-r)}$, $i = 2, 3$. At least, for the last integral in the right-hand side of (88) we have

$$
\int_\Omega \left(\int_0^{|u_n|} |D_x a(x, T_n(s))| \, ds\right) |D\phi_n^{(s)}| \leq \int_\Omega G(x)|\psi(u_n)||D\phi_n^{(s)}|.
$$

We notice that using (84) we obtain

$$
\begin{aligned}
\int_\Omega G(x)|\psi(u_n)||D\phi_n^{(s)}| \leq &\int_{\Omega_s} G(x)|\psi(u_n)||D[\psi(u_n)]_s|^{r-1} \\
&+ \int_\Omega G(x)|\psi(u_n)||H_n^{(s)}|. \qquad (91)
\end{aligned}
$$

Moreover, using (34) and (29) of Lemma 2.2, Hoelder, Young, and Sobolev inequalities it follows that if $r < N$ (that is, if $N \geq 2$) then for the first term in the right-hand side of (91) we have

$$\begin{aligned}
\int_{\Omega_s} & G(x)|\psi(u_n)||D[\psi(u_n)]_s|^{r-1} \\
&\leq \sum_{i=s}^{l} \int_{\Omega_s} G(x)|[\psi(u_n)]_i||D[\psi(u_n)]_s|^{r-1} \\
&\leq \sum_{i=s}^{l} \|G\|_{L^N(\Omega_s)} \|[\psi(u_n)]_i\|_{L^{r^*}(\Omega)} \|D[\psi(u_n)]_s\|_{L^r(\Omega_s)}^{r-1} \\
&\leq \varepsilon \sum_{i=s}^{l} c_s \|D[\psi(u_n)]_i\|_{L^r(\Omega)} \|D[\psi(u_n)]_s\|_{L^r(\Omega_s)}^{r-1} \\
&\leq \varepsilon c_s \int_{\Omega} |D[\psi(u_n)]_s|^r \\
&\qquad + \varepsilon c_s \sum_{i=s+1}^{l} \|D[\psi(u_n)]_i\|_{L^r(\Omega)} \|D[\psi(u_n)]_s\|_{L^r(\Omega_s)}^{r-1},
\end{aligned}$$

while if $N = 1$ we have

$$\begin{aligned}
\int_{\Omega_s} & G(x)|\psi(u_n)||D[\psi(u_n)]_s|^{r-1} \\
&\leq \sum_{i=s}^{l} \int_{\Omega_s} G(x)|[\psi(u_n)]_i||D[\psi(u_n)]_s|^{r-1} \\
&\leq \sum_{i=s}^{l} \|G\|_{L^r(\Omega_s)} \|[\psi(u_n)]_i\|_{L^\infty(\Omega)} \|D[\psi(u_n)]_s\|_{L^r(\Omega_s)}^{r-1} \\
&\leq \sum_{i=s}^{l} \|G\|_{L^2(\Omega_s)} (1 + |\Omega|) \|[\psi(u_n)]_i\|_{L^\infty(\Omega)} \|D[\psi(u_n)]_s\|_{L^r(\Omega_s)}^{r-1} \\
&\leq \varepsilon \sum_{i=s}^{l} c_s (1 + |\Omega|) \|D[\psi(u_n)]_i\|_{L^r(\Omega)} \|D[\psi(u_n)]_s\|_{L^r(\Omega_s)}^{r-1} \\
&\leq \varepsilon c_s (1 + |\Omega|) \int_{\Omega} |D[\psi(u_n)]_s|^r \\
&\qquad + \varepsilon c_s (1 + |\Omega|) \sum_{i=s+1}^{l} \|D[\psi(u_n)]_i\|_{L^r(\Omega)} \|D[\psi(u_n)]_s\|_{L^r(\Omega_s)}^{r-1}.
\end{aligned}$$

For the second term in the right-hand side of (91), using (85) and again Hoelder, Young, and Sobolev inequalities, it results

$$\begin{aligned}
&\int_\Omega G(x)|\psi(u_n)||H_n^{(s)}| \\
&\qquad \le c_s(1+|\Omega|)\|G\|_{L^\sigma(\Omega)}\|D\psi_n\|_{L^r(\Omega)}\|H_n^{(s)}\|_{L^{r'}(\Omega)} \\
&\qquad \le \varepsilon_4\big[c_s(1+|\Omega|)\|G\|_{L^\sigma(\Omega)}\big]^r\int_\Omega |D\psi(u_n)|^r + c(\varepsilon_4)\|H_n^{(s)}\|_{L^{r'}(\Omega)}^{r'} \\
&\qquad \le \varepsilon_4 c_G\int_{\Omega_s}|D[\psi(u_n)]_s|^r + \varepsilon_4 c_G\sum_{i=1,i\neq s}\int_{\Omega_i}|D[\psi(u_n)]_i|^r \\
&\qquad\quad + c(\varepsilon_4)[c(\Omega,N)|r-2|]^{r/(r-1)}\int_{\Omega_s}|D[\psi(u_n)]_s|^r, \qquad (92)
\end{aligned}$$

where ε_4 is a positive constant to be determined, $c(\varepsilon_4)=\varepsilon_4^{1/(1-r)}$, and $c_G=[(c_s(1+|\Omega|)\|G\|_{L^\sigma(\Omega)})^2+1]$ is a constant independent of r. Using previous estimates in (88) and recalling the choice of ε done at the beginning we obtain

$$\begin{aligned}
&\Big[1-\varepsilon_1(2+c_1(\Omega,N)|r-2|)^{r/(r-1)} \\
&\quad -\varepsilon_3-c(\varepsilon_2,\varepsilon_3,\varepsilon_4)(c(\Omega,N)|r-2|)^{r/(r-1)} \\
&\quad -\tfrac{1}{16}-\varepsilon_4 c_G\Big]\int_{\Omega_s}|D[\psi(u_n)]_s|^r \\
&\qquad \le \varepsilon_1^{1-r}\int_\Omega|F_n|^r+\varepsilon_2(l-1)^r\sum_{i=1,i\neq s}^{l}\int_{\Omega_i}|D[\psi(u_n)]_i|^r \\
&\qquad\quad +\tfrac{1}{16}\sum_{i=s+1}^{l}\|D[\psi(u_n)]_i\|_{L^r(\Omega_i)}\|D[\psi(u_n)]_s\|_{L^r(\Omega_s)}^{r-1} \\
&\qquad\quad +\varepsilon_4 c_G\sum_{i=1,i\neq s}^{l}\int_{\Omega_i}|D[\psi(u_n)]_i|^r, \qquad (93)
\end{aligned}$$

where $c(\varepsilon_2,\varepsilon_3,\varepsilon_4)=[\varepsilon_2^{1/(1-r)}+\varepsilon_3^{1/(1-r)}+\varepsilon_4^{1/(1-r)}]$. We choose $\varepsilon_1=c(\Omega,N,r)$ given by the formula

$$\varepsilon_1=\frac{1}{4[2+c_1(\Omega,N)|r-2|]^{r/(r-1)}}. \qquad (94)$$

Moreover, (67) implies the existence of a positive constant c_{30} that depends only on $\|F\|_{L^r(\Omega)}$ and is thus independent on n, such that

$$\int_\Omega|F_n|^r\le c_{30}. \qquad (95)$$

Hence from (95) and (94) it follows that

$$\varepsilon_1^{1-r} \int_\Omega |F_n|^r \leq c_{31}, \tag{96}$$

where c_{31} is a constant that depends only on Ω, N, r, and $\|F\|_{L^r(\Omega)}$. Choose $\varepsilon_3 = \frac{1}{4}$. It remains to determine ε_2 and ε_4. We impose the following restrictions for the choice of ε_4

$$\varepsilon_4 \leq \frac{1}{16 c_G} = c\big(\Omega, N, \|G\|_{L^\sigma(\Omega)}\big). \tag{97}$$

Using (94)–(97) in (93) we have

$$\begin{aligned}
&\Big[\tfrac{3}{8} - c(\varepsilon_2, r, \varepsilon_4)\big(c(\Omega, N)|r-2|\big)^{r/(r-1)}\Big] \int_{\Omega_s} \big|D[\psi(u_n)]_s\big|^r \\
&\quad \leq c_{31} + \big[\varepsilon_2 (l-1)^r + \varepsilon_4 c_G\big] \sum_{i=1, i\neq s}^{l} \int_{\Omega_i} \big|D[\psi(u_n)]_i\big|^r \\
&\qquad + \tfrac{1}{16} \sum_{i=s+1}^{l} \big\|D[\psi(u_n)]_i\big\|_{L^r(\Omega_i)} \big\|D[\psi(u_n)]_s\big\|_{L^r(\Omega_s)}^{r-1},
\end{aligned} \tag{98}$$

where $c(\varepsilon_2, r, \varepsilon_4) = [\varepsilon_2^{1/(1-r)} + 4^{1/(r-1)} + \varepsilon_4^{1/(1-r)}]$. For sake of simplicity let us assume $l = 3$; the general case is similar since by (30) it follows that $l \leq c(\Omega, N, \|G\|_{L^\sigma(\Omega)})$.

Let us choose ε_2 and ε_4 verifying, for example,

$$4\varepsilon_2 + \varepsilon_4 c_G = \frac{8^3}{(32)^3 \cdot 5 \cdot 32[1 + 1/4]}. \tag{99}$$

Notice that as $c_G > 1$, it results $\max\{\varepsilon_2, \varepsilon_4\} < 1$. Impose that r verifies the condition

$$\max\Big\{\big(c(\Omega, N)|r-2|\big)^2, c(\Omega, N)|r-2|\Big\} \leq \frac{\varepsilon_2 \varepsilon_4}{3 \cdot 4 \cdot 32}. \tag{100}$$

Thus it results

$$\kappa \equiv \frac{3}{8} - c(\varepsilon_2, r, \varepsilon_4)\big(c(\Omega, N)|r-2|\big)^{r/(r-1)} > 0. \tag{101}$$

As a matter of fact it results

$$c(\varepsilon_2, r, \varepsilon_4) \leq \frac{3}{\big((1/4)\varepsilon_2 \varepsilon_4\big)^{1/(r-1)}},$$

that together with (100) imply (101). Let $s = l = 3$ in (98): we obtain

$$\kappa \int_\Omega |D[\psi(u_n)]_3|^r \le c_{31} + [\varepsilon_2 2^r + \varepsilon_4 c_G] \sum_{i=1}^{2} \int_{\Omega_i} |D[\psi(u_n)]_i|^r. \quad (102)$$

Choosing now $s = l - 1 = 2$ in (98) we have

$$\begin{aligned} \kappa \int_\Omega |D[\psi(u_n)]_2|^r & \\ \le c_{31} + [\varepsilon_2 2^r + \varepsilon_4 c_G] & \left[\int_\Omega |D[\psi(u_n)]_1|^r + \int_\Omega |D[\psi(u_n)]_3|^r \right] \\ & + \tfrac{1}{16} \|D[\psi(u_n)]_3\|_{L^r(\Omega)} \|D[\psi(u_n)]_2\|_{L^r(\Omega)}^{r-1}. \end{aligned} \quad (103)$$

Using the Young inequality we have that the last term in the right-hand side of (103) can be estimated from above with

$$\tfrac{1}{16} \left[\int_\Omega |D[\psi(u_n)]_3|^r + \int_\Omega |D[\psi(u_n)]_2|^r \right].$$

Thus it follows, using (102), that

$$\begin{aligned} \left(\kappa - \tfrac{1}{16}\right) \int_\Omega |D[\psi(u_n)]_2|^r & \\ \le c_{31} + [\varepsilon_2 2^r + \varepsilon_4 c_G] \int_\Omega |D[\psi(u_n)]_1|^r & \\ + \left[\varepsilon_2 2^r + \varepsilon_4 c_G + \tfrac{1}{16}\right] \int_\Omega |D[\psi(u_n)]_3|^r & \\ \le c_{31} + [\varepsilon_2 2^r + \varepsilon_4 c_G] \int_\Omega |D[\psi(u_n)]_1|^r & \\ + \frac{\varepsilon_2 2^r + \varepsilon_4 c_G + \frac{1}{16}}{\kappa} \Bigg\{ c_{31} + [\varepsilon_2 2^r + \varepsilon_4 c_G] & \\ \times \left[\int_{\Omega_1} |D[\psi(u_n)]_1|^r + \int_\Omega |D[\psi(u_n)]_2|^r \right] \Bigg\}. & \end{aligned} \quad (104)$$

Notice that our choices imply $\kappa - \frac{1}{16} > 0$. Now we need "to absorb" in the left-hand side the last integral in the right-hand side. To ensure this, if we define

$$\kappa_1 \equiv \kappa - \tfrac{1}{16} = \tfrac{5}{16} - c(\varepsilon_2, r, \varepsilon_4)(c(\Omega, N)|r - 2|)^{r/(r-1)}, \quad (105)$$

it is sufficient to prove that

$$\begin{cases} \kappa_1 - \dfrac{\varepsilon_2 2^r + \varepsilon_4 c_G + 1/16}{\kappa}\left[\varepsilon_2 2^r + \varepsilon_4 c_G\right] \geq \kappa_1 - \dfrac{1}{26}, \\ \kappa_2 \equiv \kappa_1 - \dfrac{1}{26} > 0, \end{cases} \tag{106}$$

that is,

$$\begin{cases} \kappa \geq 26\left[\varepsilon_2 2^r + \varepsilon_4 c_G + \frac{1}{16}\right]\left[\varepsilon_2 2^r + \varepsilon_4 c_G\right], \\ \kappa_2 > 0. \end{cases} \tag{107}$$

Notice that (107) is verified because for our choice of ε_2 and ε_4 it results

$$32\left[4\varepsilon_2 + \varepsilon_4 c_G + \tfrac{1}{16}\right]\left[4\varepsilon_2 + \varepsilon_4 c_G\right] \leq \tfrac{1}{16}, \tag{108}$$

and because r satisfies the restriction

$$\left[\varepsilon_2^{1/(1-r)} + 4^{1/(r-1)} + \varepsilon_4^{1/(1-r)}\right]\left(c(\Omega, N)|r-2|\right)^{r/(r-1)} < \tfrac{9}{32}, \tag{109}$$

since we have

$$\begin{cases} \max\left\{\left(c(\Omega, N)|r-2|\right)^2, c(\Omega, N)|r-2|\right\} \leq \dfrac{27\varepsilon_2\varepsilon_4}{4 \cdot 32}, \\ \max\{\varepsilon_2, \varepsilon_4\} \leq 1. \end{cases} \tag{110}$$

Thus we obtain

$$\begin{aligned} \kappa_2 \int_\Omega \left|D[\psi(u_n)]_2\right|^r \leq c_{31}&\left(1 + \frac{\varepsilon_2 2^r + \varepsilon_4 c_G + \frac{1}{16}}{\kappa}\right) \\ &+ \left[\varepsilon_2 2^r + \varepsilon_4 c_G\right]\left(1 + \frac{\varepsilon_2 2^r + \varepsilon_4 c_G + \frac{1}{16}}{\kappa}\right) \\ &\cdot \int_\Omega \left|D[\psi(u_n)]_1\right|^r. \end{aligned} \tag{111}$$

In order to simplify (111) we observe that by (108) it follows that

$$\varepsilon_2 2^r + \varepsilon_4 c_G + \frac{1}{16} \leq \varepsilon_2 2^2 + \varepsilon_4 c_G + \frac{1}{16} \leq 1 \leq \frac{1}{\kappa}. \tag{112}$$

Hence using (112) in (111) we obtain

$$\kappa_2 \int_\Omega \left| D[\psi(u_n)]_2 \right|^r \le \frac{2c_{31}}{\kappa} + \frac{2[\varepsilon_2 2^r + \varepsilon_4 c_G]}{\kappa} \int_\Omega \left| D[\psi(u_n)]_1 \right|^r. \quad (113)$$

Using (113) and (112) in (102) we obtain an estimate of $\int_\Omega |D[\psi(u_n)]_3|^r$ only in dependence of $\int_\Omega |D[\psi(u_n)]_1|^r$ that is

$$\kappa \int_\Omega \left| D[\psi(u_n)]_3 \right|^r \le c_{31}\left[1 + \frac{2}{\kappa\kappa_2}\right] + [\varepsilon_2 2^r + \varepsilon_4 c_G]\left[1 + \frac{2}{\kappa\kappa_2}\right] \int_{\Omega_1} \left| D[\psi(u_n)]_1 \right|^r. \quad (114)$$

To conclude choose $s = 1$ in (98) and apply the Young inequality in the last term in the right-hand side. Hence we obtain

$$\kappa \int_{\Omega_1} \left| D[\psi(u_n)]_1 \right|^r \le c_{31} + [\varepsilon_2 2^r + \varepsilon_4 c_G] \sum_{i=2}^{3} \int_{\Omega_i} \left| D[\psi(u_n)]_i \right|^r + \frac{1}{16}\left[\left(\sum_{i=2}^{3} \| D[\psi(u_n)]_i \|_{L^r(\Omega_i)} \right)^r + \int_\Omega \left| D[\psi(u_n)]_1 \right|^r \right], \quad (115)$$

from which we have

$$\kappa_1 \int_{\Omega_1} \left| D[\psi(u_n)]_1 \right|^r \le c_{31} + [\varepsilon_2 2^r + \varepsilon_4 c_G] \sum_{i=2}^{3} \int_{\Omega_i} \left| D[\psi(u_n)]_i \right|^r + \frac{2^r}{16} \sum_{i=2}^{3} \int_\Omega \left| D[\psi(u_n)]_i \right|^r. \quad (116)$$

Using the estimate (112)–(114), it results

$$
\begin{aligned}
\kappa_1 \int_\Omega & \left| D[\psi(u_n)]_1 \right|^r \\
& \leq c_{31} + \left[\varepsilon_2 2^r + \varepsilon_4 c_G + \frac{2^r}{16} \right] \left(\int_\Omega \left| D[\psi(u_n)]_2 \right|^r + \int_\Omega \left| D[\psi(u_n)]_3 \right|^r \right) \\
& \leq c_{31} \left[1 + \left(1 + \frac{2^r}{16} \right) \frac{5}{\kappa^2 \kappa_2} \right] \\
& \quad + [\varepsilon_2 2^r + \varepsilon_4 c_G] \left[1 + \frac{2^r}{16} \right] \frac{5}{\kappa^2 \kappa_2} \int_\Omega \left| D[\psi(u_n)]_1 \right|^r,
\end{aligned}
\tag{117}
$$

as $(1/\kappa)(1 + 2/\kappa\kappa_2) + 2/\kappa\kappa_2 \leq 5/\kappa^2\kappa_2$. We can absorb the last integral of the right-hand side in the left-hand side as the following inequality holds

$$
\kappa_1 - [\varepsilon_2 2^r + \varepsilon_4 c_G] \left[1 + \frac{2^r}{16} \right] \frac{5}{\kappa^2 \kappa_2} \geq \kappa_1 - \frac{1}{32}.
$$

As a matter of fact, the previous inequality follows from the estimate

$$
4\varepsilon_2 + \varepsilon_4 c_G \leq \frac{\kappa_2^3}{5 \cdot 32[1 + 1/4]}. \tag{118}
$$

To prove (118) it is sufficient to recall the choice done for ε_2 and ε_4 and to observe that from the choice done for r it follows that

$$
\kappa_2 \geq \tfrac{8}{32}, \tag{119}
$$

as it results

$$
\left[\varepsilon_2^{1/(1-r)} + 4^{1/(r-1)} + \varepsilon_4^{1/(1-r)} \right] \left(c(\Omega, N) |r - 2| \right)^{r/(r-1)} \leq \tfrac{1}{32}. \tag{120}
$$

Thus we conclude that there exists a positive constant c, independent on n, such that

$$
\int_\Omega \left| D[\psi(u_n)]_1 \right|^r \leq c. \tag{121}
$$

Using (121) in (113) and (114) we obtain that for every $i = 1, 2, 3$ the following estimate holds

$$
\int_\Omega \left| D[\psi(u_n)]_i \right|^r \leq c_{32}, \tag{122}
$$

where c_{32} is a constant depending only on $\|G\|_{L^\sigma(\Omega)}$, r, $\|F\|_{L^r(\Omega)}$, Ω, and N, as by (30) it follows that $l \le c(\Omega, N, \|G\|_{L^\sigma(\Omega)})$. From (33) and (122) it follows (83) which implies (82) proceeding as in Lemma 3.4 (see formula (65)). Q.E.D.

4. PROOFS OF THEOREMS 1.1–1.5

Proof of Theorem 1.1. Let u_n be the solution of problem (35). By Lemma 3.3 and the Sobolev embedding theorem, passing to a subsequence, that we denote again with u_n, we have

$$\begin{aligned} &Du_n \rightharpoonup Du && \text{weakly in } L^q(\Omega),\\ &u_n \to u && \text{in } L^\lambda(\Omega),\\ &T_n(u_n) \to u && \text{in } L^\lambda(\Omega),\\ &a(x, T_n(u_n)) \to a(x,u) && \text{in } L^p(\Omega), \qquad \forall p \in [1, +\infty), \end{aligned} \tag{123}$$

for every $\lambda \in [1, q^*)$ if $N \ge 2$ and for every $\lambda \in [1, +\infty)$ if $N = 1$. Thanks to (123) we can pass to the limit as $n \to +\infty$ in (35) and we obtain that u is a solution of problem (6).

In order to prove (11) we prove now that $T_k(u_n)$ is bounded in $H_0^1(\Omega)$ by a constant $c = c(k, \|F\|_{L^2})$ independent on n. Let us take $T_k(u_n)$ as a test function in (35). We obtain using assumption (7)

$$\frac{c_0}{(1+k)^\theta}\int_\Omega |DT_k(u_n)|^2 \le \|F\|_2\left(\int_\Omega |DT_k(u_n)|^2\right)^{1/2},$$

and thus we have

$$\|DT_k(u_n)\|_{L^2(\Omega)} \le \frac{1+k}{c_0}\|F\|_2. \tag{124}$$

Hence it follows that

$$DT_k(u_n) \rightharpoonup DT_k(u) \qquad \text{weakly in } L^2(\Omega) \text{ as } n \to \infty. \tag{125}$$

We observe that from

$$\int_\Omega \frac{1}{(1+|T_n(u_n)|)^{2\theta}}|D(T_k(u_n) - T_k(u))|^2 \ge 0,$$

it follows that

$$2\int_{\Omega} \frac{1}{(1+|T_n(u_n)|)^{2\theta}} DT_k(u_n)\, DT_k(u) - \int_{\Omega} \frac{1}{(1+|T_n(u_n)|)^{2\theta}} |DT_k(u)|^2$$
$$\leq \int_{\Omega} \frac{|DT_k(u_n)|^2}{(1+|T_n(u_n)|)^{2\theta}}. \tag{126}$$

Now using (123), (125), and (38) of Lemma 3.3 we deduce that

$$\int_{\Omega} \frac{|DT_k(u)|^2}{(1+|u|)^{2\theta}} \leq c,$$

from which by Fatou's Lemma we obtain (11).

Proof of Theorem 1.2. Let u_n be the solution of problem (35). Let us choose as a test function $T_k(u_n - \varphi)$ where $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We obtain

$$\int_{\Omega} a(x, T_n(u_n))\, Du_n\, DT_k(u_n - \varphi) = \int_{\Omega} F\, DT_k(u_n - \varphi).$$

We notice that (124) holds also in this case. Hence it is possible to proceed exactly as in the proof of Theorem 1.17 in [3] and conclude that there exists an entropy solution of (6). The proof of (13) follows from (126), (125), and (38) of Lemma 3.3 (which holds for every $0 < \theta < 1$ fixed) noticing that if $n > k$ then it results $T_n(u_n) = T_k(u_n)$ in the set $\{|u_n| \leq k\}$. Q.E.D.

Proof of Theorem 1.3. Let u_n be the solution of problem (35). In order to prove (20) it is sufficient to follow the outline of the proof of (11) in Theorem 1.1 and (13) in Theorem 1.2 changing the term $1/(1+|T_n(u_n)|)^\theta$ with $a(x, T_n(u_n))$ and using (47) of Lemma 3.4 (which holds for every $0 < \theta < 1$ fixed) instead of (38) of Lemma 3.3. Q.E.D.

Proof of Theorem 1.4. Let u_n be the solution of problem (66). By Lemma 3.5 if q is as in (24) and $N \geq 2$, thus it results $q < r$ and we have, proceeding exactly as in the proof of Theorem 1.3,

$$
\begin{aligned}
c_s^{-q}\left(\int_\Omega |u_n|^{q^*}\right)^{q/q^*} &\le \int_\Omega |Du_n|^q \\
&\le \left(\int_\Omega \frac{|Du_n|^r}{(1+|u_n|)^{\theta r}}\right)^{q/r}\left(\int_\Omega (1+|u_n|)^{\theta q (r/q)'}\right)^{1-q/r} \\
&\le c_{25}^{q/r} 2^{q^*}\left[\left(\int_\Omega |u_n|^{q^*}\right)^{1-q/r} + |\Omega|^{1-q/r}\right], \qquad (127)
\end{aligned}
$$

as $\theta q(\frac{r}{q})' = q^*$. Noticing that $1 - q/r < q/q^*$ if $r < N$, from (127) we obtain that if $N \ge 2$ then it results

$$\int_\Omega |u_n|^{q^*} \le c, \qquad (128)$$

where c is a constant independent on n. Again using (128) in (127) we obtain the following a priori estimate

$$\int_\Omega |Du_n|^q \le c. \qquad (129)$$

If otherwise $N = 1$ proceeding as before, but using the immersion of $W_0^{1,r}(\Omega)$ in $L^\infty(\Omega)$ the estimate (128) is replaced by

$$\|u_n\|_{L^\infty(\Omega)} \le c, \qquad (130)$$

where c is as in (128) and again we can deduce that (129) holds. Now we can conclude the proof proceeding as in the proof of Theorem 1.1. Q.E.D.

Proof of Theorem 1.5. The proof of Theorem 1.5 make use of Lemma 3.6 and is similar to the previous one. Hence we omit it.

REFERENCES

1. P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and L. Vasquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **22** (1995), 241–273.
2. L. Boccardo, Quelques problèmes de Dirichlet avec données dans de grands espaces de Sobolev, *C. R. Acad. Sci. Paris Sér. I* **325** (1971), 1269–1272.
3. L. Boccardo, A. Dall'Aglio, and L. Orsina, Existence and regularity results for some elliptic equations with degenerate coercivity, *Atti Sem. Math. Fis. Univ. Modena* **46** (1998), 51–81.

4. T. Del Vecchio and M. M. Porzio, Existence results for a class of non coercive Dirichlet problems, *Ric. Mat.* **44** (1995), 421–438.
5. T. Iwaniec, p-Harmonic tensors and quasiregular mappings, *Ann. of Math.* **136** (1992), 589–624.
6. T. Iwaniec and C. Sbordone, Weak minima of variational integrals, *J. Reine Angew. Math.* **454** (1994), 143–161.
7. J. L. Lions, "Quelques méthods de resolution de problèmes aux limites non linéaires," Dunod, Gauthier–Villars, Paris, 1969.
8. N. G. Meyers, An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 189–206.
9. A. Porretta, Uniqueness and homogenization for a class of non coercive operators in divergence form, *Atti Sem. Mat. Fis. Univ. Modena* **46** (1998), 915–936.